A Note on Weighted Rooted Trees

Zi-Xia Song*, Talon Ward and Alexander York
Department of Mathematics
University of Central Florida
Orlando, FL 32816, USA

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Abstract

Let T be a tree rooted at r. Two vertices of T are *related* if one is a descendant of the other; otherwise, they are *unrelated*. Two subsets A and B of $V(T)$ are *unrelated* if, for any $a \in A$ and $b \in B$, a and b are unrelated. Let ω be a nonnegative weight function defined on $V(T)$ with $\sum_{v\in V(T)} \omega(v) = 1$. In this note, we prove that either there is an (r, u)-path P with $\sum_{v\in V(P)} \omega(v) \geq \frac{1}{3}$ for some $u \in V(T)$, or there exist unrelated sets $A, B \subseteq V(T)$ such that $\sum_{a\in A} \omega(a) \geq \frac{1}{3}$ and $\sum_{b\in B} \omega(b) \geq \frac{1}{3}$. The bound $\frac{1}{3}$ is tight. This answers a question posed in a very recent paper of Bonamy, Bousquet and Thomassé.

1 Introduction

Let T be a tree rooted at r. Let $x \in V(T)$. A *descendant* of x is any vertex y such that $x \in V(P)$, where P is the unique (r, y)-path in T. The *parent* of x is the vertex y such that y immediately precedes x on the unique (r, x)-path in T. Two vertices of T are *related* if one is a descendant of the other; otherwise, they are *unrelated*. Two subsets A and B of $V(T)$ are *unrelated* if, for any $a \in A$ and $b \in B$, a and b are unrelated. Note that if A and B are unrelated, then $A \cap B = \emptyset$. Let G be a graph and let ω be a nonnegative weight function defined on $V(G)$. For any $A \subseteq V(G)$ and any subgraph H of G, define $\omega(A) := \sum_{a\in A} \omega(a)$ and $\omega(H) = \omega(V(H))$. In their proof of the main result in [1], Bonamy, Bousquet and Thomassé made use of the following lemma.

Lemma 1.1 *Let T be a tree rooted at r and let ω be a nonnegative weight function defined on $V(T)$ with $\omega(T) = 1$. Then there is an (r, u)-path P with $\omega(P) \geq \frac{1}{4}$ for some $u \in V(T)$, or there exist unrelated sets $A, B \subseteq V(T)$ such that $\omega(A) \geq \frac{1}{4}$ and $\omega(B) \geq \frac{1}{4}$.*

*Corresponding Author: Zixia.Song@ucf.edu

In the same paper, the authors believe that Lemma 1.1 holds for $\frac{1}{3}$. This problem has a Ramsey Theory flavor. In this note, we give an affirmative answer to their question and point out that the bound $\frac{1}{3}$ is tight.

Theorem 1.2 *Let T be a tree rooted at r and let ω be a nonnegative weight function defined on $V(T)$ with $\omega(T) = 1$. Then there is an (r, u)-path P with $\omega(P) \geq \frac{1}{3}$ for some $u \in V(T)$, or there exist unrelated sets $A, B \subseteq V(T)$ such that $\omega(A) \geq \frac{1}{3}$ and $\omega(B) \geq \frac{1}{3}$. The bound $\frac{1}{3}$ is tight.*

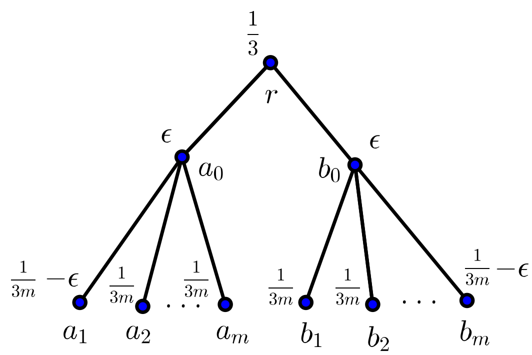

Figure 1: Rooted tree T.

To see why the bound $\frac{1}{3}$ is best possible. Let $m \geq 2$ be an integer and $\epsilon \geq 0$ be a small number with $\epsilon \leq \frac{1}{3m}$. Let T be the weighted tree rooted at r as depicted in Figure 1. Note that ω is a positive weight function on $V(T)$ when $\frac{1}{3m} > \epsilon > 0$. Any path from the root r in T has weight between $\frac{1}{3}$ and $\frac{1}{3} + \frac{1}{3m} + \epsilon$; and T has one unique pair of unrelated sets $A = \{a_0, a_1, a_2, \ldots, a_m\}$ and $B = \{b_0, b_1, b_2, \ldots, b_m\}$ with $\omega(A) = \omega(B) = \frac{1}{3}$. The bound $\frac{1}{3}$ is tight when m is large.

2 Proof of Theorem 1.2

Suppose T has no path from the root r with weight at least 1/3. Then T is not a path. Let $N_G(r) = \{v_1, v_2, \ldots, v_s\}$ and $T_1, T_2, \ldots, T_s$ be connected components of $T - r$, where $\omega(T_1) \leq \omega(T_2) \leq \cdots \leq \omega(T_s)$. We call each T_i a subtree of T rooted at v_i for $1 \leq i \leq s$. And $T_1, \ldots, T_s$ are also called *branches* of T at r. We next construct two unrelated sets A and B with desired weights according to the following algorithm:

Algorithm 1: Building Sets A and B

Data: Vertex weighted tree T with root r

Result: Unrelated sets A and B with desired weights

Start at the root r with $A = B = \emptyset$ and set $C = \{T_1, T_2, \dots, T_s\}$;

while $C \neq \emptyset$ **do**

  for $i = 1$ *to* $s-1$ **do**

    Remove T_i from C. Add the vertices of T_i to A if $\omega(A) \leq \omega(B)$, and to B otherwise;

  end

  If $\emptyset \neq \bigcup_{i=1}^{s-1} V(T_i) \subseteq A$ (resp. $\emptyset \neq \bigcup_{i=1}^{s-1} V(T_i) \subseteq B$), color the root r RED (resp. BLUE), otherwise color the root r GREEN ;

  Set r to be the root of T_s and C be the set of connected components of $T_s \backslash r$ with weights sorted in the nondecreasing order;

end

Call the last root r^*. If $\omega(A) \leq \omega(B)$, add r^* to A and color r^* RED, otherwise add r^* to B and color r^* BLUE. Let $y = r^*$, x be the parent of y and c be the color of y;

while x *is colored GREEN or* c **do**

  re-color x by the color c if x is colored GREEN and $d_T(x) = 2$;

  Set y to be x, and x be the parent of y ;

end

Let $u = r^*$;

while u *is adjacent to a vertex* $v \notin A \cup B$ *with the same color as* r^* **do**

  Add v to A if both u and v are colored RED, and add v to B if both u and v are colored BLUE;

  Set u to be v;

end

It can be easily checked that A and B constructed by the above algorithm are unrelated. Since T is not a path, both A and B are nonempty. Let u be the vertex in the last step of the algorithm that is added to A or B. According to the algorithm, u is colored RED or BLUE. Let M be the set of all colored vertices of T. Then the subgraph $T[M]$ of T induced by M is the unique (r, r^*)-path, say P, where r^* is the last root as given in the algorithm. By the algorithm, $T - A \cup B$ is the unique (r, u^*)-path, say P^*, of T, where u^* is the parent of u in T. Clearly, P^* is a subpath of P. Let $N = V(P) - V(P^*)$. Then $r^* \in N$ and the vertices of N are all colored by the same color of the root r^*. One can see that if u is colored RED, then $u \in N \subseteq A$ and the last set of vertices added to B are all uncolored. Similarly, if u is colored BLUE, then $u \in N \subseteq B$ and the last set of vertices added to A are all uncolored. Since $\omega(P^*) + \omega(A) + \omega(B) = 1$ and $\omega(P^*) < \frac{1}{3}$, we have

(1) $\omega(A) + \omega(B) > \frac{2}{3}$.

We next show that $\min\{\omega(A), \omega(B)\} \geq \frac{1}{3}$.

Suppose that $\omega(A) \leq \omega(B)$. By (1), $\omega(B) \geq \frac{1}{3}$. Assume $u \in B$. Then u is colored BLUE and so r^* is also colored BLUE. Thus $N \subseteq B$. Since r^* is added to B, we have $\omega(A) \geq \omega(B - N)$. On the other hand, $\omega(A) + \omega(B - N) = 1 - \omega(P) > \frac{2}{3}$. Thus $\omega(A) > \frac{1}{3}$, as desired. So we may assume $u \in A$. Then u is colored RED and so $N \subseteq A$. Let D be the set of vertices that were last added to B. Then D contains only uncolored vertices of T. Thus $D = V(Y)$, where Y is a branch of some subtree T^* of T. Since D contains only uncolored vertices, by the algorithm, T^* has a branch X with $\omega(Y) \leq \omega(X)$ and $X \cap B = \emptyset$. Let X^* be the set of all vertices that are added to A after the vertices in D were added to B. By the algorithm, $X \subseteq X^*$, and so $\omega(X^*) \geq \omega(X) \geq \omega(Y)$. Let $\tilde{A} = A - X^*$ and $\tilde{B} = B - Y$. Since Y is added to B, we have $\omega(\tilde{A}) \geq \omega(\tilde{B})$. Note that $\omega(A) = \omega(\tilde{A}) + \omega(X^*)$ and $\omega(B) = \omega(\tilde{B}) + \omega(Y)$. Thus $\omega(B) \leq \omega(\tilde{A}) + \omega(X^*) = \omega(A)$. By (1), $\omega(A) \geq \frac{1}{3}$. Hence $\frac{1}{3} \leq \omega(A) \leq \omega(B)$, as desired.

By a similar argument as above, one can show that $\min\{\omega(A), \omega(B)\} \geq \frac{1}{3}$ for the case when $\omega(B) \leq \omega(A)$. This completes the proof of Theorem 1.2. ■

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References

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